OPTIMIZING TELEPORTATION PROTOCOLS

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ABSTRACT

In this project, we study quantum teleportation protocols with noisy resources using variational quantum optimization (VQO). Quantum teleportation is a fundamental quantum information-theoretic task in which Alice aims to teleport an unknown quantum state to Bob using a shared entanglement resource and classical communication. A teleportation protocol consists of a measurement implemented by Alice, a classical channel transmitting the measurement outcome to Bob, and a set of correction operations implemented by Bob depending on the measurement outcome. For a maximally entangled state, the well-known standard teleportation protocol by Bennett et al. [1] defined in terms of a Bell measurement and Pauli corrections gives a perfect protocol. However, in the presence of noise such a perfect teleportation protocol is generally impossible, and instead one aims to maximize the so-called teleportation fidelity of a protocol by finding suitable measurements and correction operations. Here, we use a VQO ansatz simulated in the PennyLane framework in order to find teleportation protocols achieving non-classical fidelities for noisy entangled resource states. We carry out a detailed numerical study of teleportation protocols with both unitary and noisy elements for the class of Badziag et. al states, which are mixtures of two weighted Bell states. Furthermore, we examine qutrit-Werner states and ququart-Werner states, representing a spectrum of mixtures of fully mixed and maximally entangled states within a three-level or four-level quantum system, for usefulness as entangled resources in teleportation protocols.

Subject Keywords: Variational Quantum Optimization (VQO); Quantum Network Variational Optimization (qNetVO); Quantum Optimization; Quantum Teleportation; Dense Coding; Noisy Quantum Channels

TABLE OF CONTENTS

INTRODUCTION

1.1 Quantum teleportation

Quantum teleportation is one of the most important protocols in quantum information theory, offering a novel method for transmitting quantum information. This protocol is the backbone of several far-reaching applications, including long-distance entanglement distribution through repeaters [2], secret key distribution for quantum cryptography [3], fusion-based quantum computing [4], secret sharing [5], entanglement swapping [2], supplanting conventional connections in large quantum computers, universal distributed quantum computation, and the quantum internet [6].

At its core, a teleportation protocol dismantles an unknown quantum state into classical information and entanglement, a nonclassical correlation, allowing for its reconstruction over a distance. Quantum teleportation demonstrates the operational interchangeability of one communication primitive, a quantum channel, with two other communication primitives, entanglement and a classical channel. Essentially, quantum teleportation emulates quan-

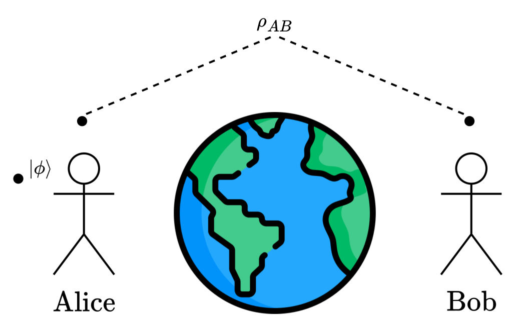


Figure 1.1: A teleportation protocol begins with two parties, Alice and Bob, where Alice wishes to transmit the unknown state $|\phi\rangle$ to Bob. In order to perform teleportation, Alice and Bob must share a bipartite quantum state ρ_{AB} which is entangled.

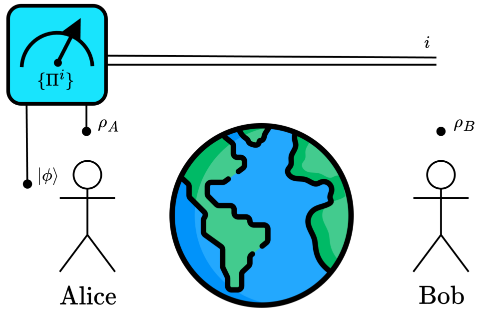


Figure 1.2: Teleportation highlights the relationship between one communication primitive, transmission of quantum information, and two others, entanglement and transmission of classical information. As part of the protocol, Alice can disassemble her unknown quantum state $|\phi\rangle$ into classical information i and nonclassical correlations from ρ_{AB}.

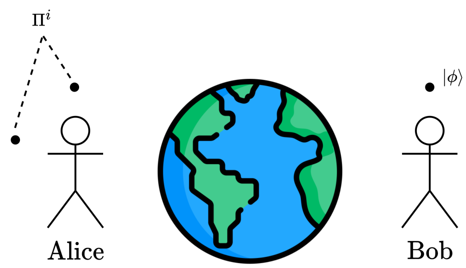


Figure 1.3: Alice's quantum state $|\phi\rangle$ is deposited in Bob's quantum particle at the end of teleportation. Any entanglement between Alice and Bob is gone, and Alice's particles retain no trace of any information about $|\phi\rangle$. The protocol is limited by the speed of light because Bob must apply a correction operation based on the classical message i. If Bob does not wait for Alice's classical message and use it, then he instead obtains the maximally mixed state.

tum communication through the consumption of shared entanglement, application of local quantum operations, and use of classical communication.

Using teleportation, two parties Alice and Bob spend the entanglement of a shared quantum resource to reliably transmit an unknown quantum state across physically distant quantum systems. Despite the name quantum teleportation, nothing is literally being "teleported." The quantum state is decomposed into classical information and non-classical correlations by joint measurement with Alice's part of the entangled resource, the measurement results are communicated from Alice to Bob, and the quantum state is recreated through state transformation. At no point in the protocol is the no-cloning theorem violated, that is, the unknown quantum state is not copied or doubled. In section 2.1, we discuss the theory of teleportation and delve

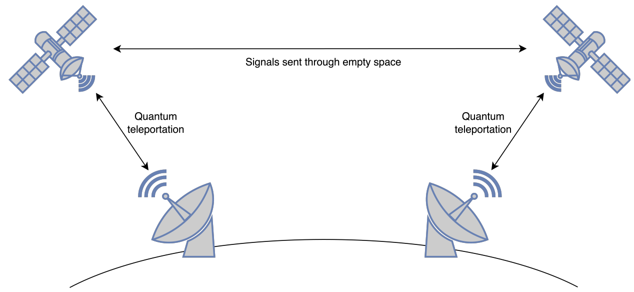


Figure 1.4: Satellites can help connect distant local quantum networks. Ground stations utilize quantum teleportation to send quantum information to satellites. This is achieved by exploiting entanglement between the ground station and satellite. An encoding measurement is performed at the ground station, and the unknown quantum state can be recreated at the satellite using classical information sent from the ground. Satellites take advantage of the low-error, low-decoherence environment provided by space to communicate quantum information directly with one another.

into the equivalence of communication primitives.

The anticipation for advanced applications of quantum communication is growing. For example, BT and Toshiba's pioneering quantum-secured metro network [7] currently showcases a secure link between a large bank's headquarters and a nearby data centre using quantum key distribution. The commercial viability of quantum networking is unknown but promising, and it is not far-fetched to expect more complex activities in the future.

Quantum teleportation plays a key role in quantum communication as systems scale. The quantum-secured metro network in London utilizes fibre optic cables. However, strong attenuation factors in fibre optic cables prevent the reliable transmission of quantum information over long distances. Classical information can be duplicated to statistically guarantee reliable communication across fibre optic cables, but the no-cloning theorem of quantum information prevents the duplication of quantum information. Thus, the quantum teleportation protocol emerges as a natural solution for sending quantum information across larger distances [2]. Previous work has already implemented ground-to-satellite quantum teleportation in hopes for a large-scale quantum network [8, 9] 1.4. Industry is already adopting these techniques, as Toshiba is building a terrestrial QKD solution with satellite technology [10].

For a maximally entangled shared resource, the standard teleportation protocol by Bennett et al. [1] defined in terms of a Bell measurement and Pauli corrections gives a perfect protocol with flawless information transfer. In the presence of noise such a perfect teleportation protocol is general impossible, and instead one aims to maximize the so-called teleportation fidelity of a protocol by finding suitable measurements and correction operations. The average fidelity of a quantum teleportation protocol is the average overlap between target and teleported states, and is the quantity we seek to maximize in real teleportation protocols.

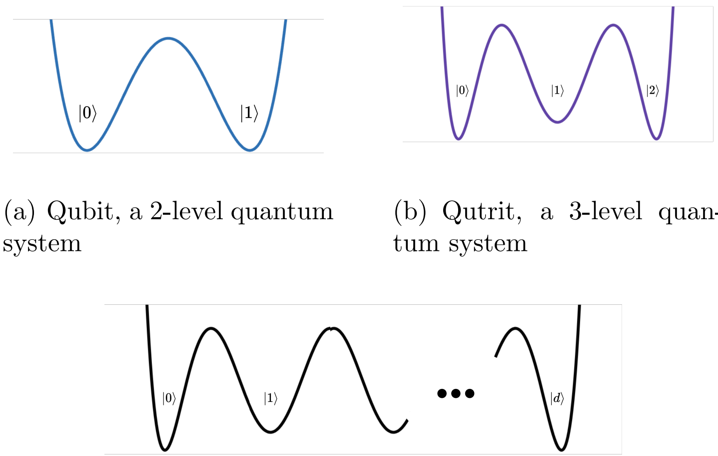
A common noisy experimental scenario is when the entangled resource state is a Werner state produced by the action of a depolarizing channel acting on one part of a singlet state, a maximally entangled state. In that case, the standard teleportation protocol is no longer optimal. This work efficiently discovers high-dimensional Werner states for which a teleportation protocol with non-classical fidelity exists. Furthermore, this work parameterizes the quantum teleportation protocol to efficiently optimize teleportation in the presence of any noise, improving the quality of quantum communication.

1.2 Qudits

The most popular unit of quantum information is the qubit, a 2-level quantum system. However, d-level quantum systems known as qudits have recently garnered attention for their increased information capacity [11] and potential for robust quantum communication [12].

The entanglement properties of a system with two qubits is more-or-less well understood. Necessary and sufficient criterion for identifying entanglement in qubit-qubit and even qubit-qutrit systems is computationally simple [13, 14]. However, there is not yet any such characterization of entanglement for much larger systems. Uncovering the utility of d-level bipartite quantum systems for entanglement-requiring tasks like teleportation is a rich and complicated question.

This work focuses on optimizing teleportation protocols with various families of noisy entangled resource states, including 3-dimensional and 4-dimensional qudits (qutrits and ququarts). We numerically demonstrate the range of states for which certain families of states provide a quantum advantage for teleportation at different dimensions. This work develops and utilizes an efficient framework for characterizing a quantum state's utility for noisy teleportation protocols.



(a) Qubit, a 2-level quantum system

(b) Qutrit, a 3-level quantum system

(c) Qudit, a d-level quantum system

Figure 1.5: Units of quantum information

1.3 Variational quantum optimization

Variational quantum optimization (VQO) is a technique for splitting large computational loads between classical and quantum computers for optimization tasks. This technique is applicable to the current NISQ era where available runs on large quantum computers are slim. VQO employs parameterized quantum circuits and optimizes them according to a problem-specific cost function. Classical computational feedback is compute the objective function in a feedback loop. We use VQO to optimize teleportation protocols and identify quantum states which would provide a nonclassical advantage for teleportation.

1.4 Main contribution

The main contributions of this paper include

- Using the correspondence between teleportation and dense coding to efficiently optimize teleportation protocols given a noisy entangled resource.

- Carrying out a detailed numerical study of teleportation protocols with both unitary and noisy elements for the class of Bazdiag et. al states, which are mixtures of two weighted Bell states.

- Efficient discovery of 3-level and 4-level Werner states which are useful for teleportation through a surprisingly efficient calculation of the reduction criterion for versions of these states that are locally processed on only one side.

THEORETICAL BACKGROUND

2.1 Teleportation

Teleportation simulates a quantum channel with communication primitives which are better suited for long-distance communication: shared entanglement and a classical channel. A general teleportation protocol from Alice to Bob consists of:

- An entangled state, ρ_{AB}
- Encoding measurement, $\{\Pi^i\}$
- Decoding state transformation, $\{D^i\}$

2.1.1 Teleporting qubits

An unknown qubit state $|\phi\rangle$ can be dissassembled into, then later reconstructed from, purely classical information and purely nonclassical Einstein-Podolsky-Rosen (EPR) correlations [1].

Suppose that Alice wishes to transfer the state $|\phi\rangle$ of her qubit (particle 1) to Bob. The following protocol allows Alice to flawlessly transmit $|\phi\rangle$ to Bob without a physical quantum connection.

In the standard teleportation protocol for qubits, the entangled resource state is an EPR singlet state. This is a maximally entangled bipartite state of qubits. One EPR particle (particle 2) is given to Alice, while the other (particle 3) is given to Bob.

$$\rho_{AB} = |\Psi_{23}^{(-)}\rangle = \frac{1}{\sqrt{2}}(|0_2\rangle|1_3\rangle - |1_2\rangle|0_3\rangle)$$

The encoding measurement entangles the unknown quantum particle $|\phi_1\rangle$ with the entangled pair. The encoding measurement is performed in the bell operator basis, given by $|\Psi_{12}^{(\pm)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle)$ and $|\Phi_{12}^{(\pm)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle)$.

$$\{\Pi^i\} = \{|\Psi^{(-)}\rangle, |\Psi^{(+)}\rangle, |\Phi^{(-)}\rangle\, |\Phi^{(+)}\rangle\}$$

Note that $|\phi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha^2 + \beta^2 = 1$. The complete state of the three particles before Alice's measurement is then

$$\begin{aligned}
|\Psi_{123}\rangle = |\phi_1\rangle \otimes |\Psi_{23}^{(-)}\rangle &= \frac{\alpha}{\sqrt{2}}(|0_1\rangle|0_2\rangle|1_3\rangle - |0_1\rangle|1_2\rangle|0_3\rangle) + \frac{\beta}{\sqrt{2}}(|1_1\rangle|0_2\rangle|1_3\rangle - |1_1\rangle|1_2\rangle|0_3\rangle) \\
&= \frac{1}{2}\Big[|\Psi_{12}^{(-)}\rangle(-\alpha|0_3\rangle - \beta|1_3\rangle) + |\Psi_{12}^{(+)}\rangle(-\alpha|0_3\rangle + \beta|1_3\rangle) \\
&\quad + |\Phi_{12}^{(-)}\rangle(\alpha|1_3\rangle + \beta|0_3\rangle) + |\Phi_{12}^{(+)}\rangle(\alpha|1_3\rangle - \beta|0_3\rangle)\Big]
\end{aligned}$$

Thus, the decoding state transformation is a simple function of Alice's measurement outcome. Measuring $|\Psi_{12}^{(-)}\rangle$ suggests that Bob's state has been projected into $(-\alpha|0\rangle - \beta|1\rangle)$. In this case, Bob's state is the same as $|\phi\rangle$ aside from an irrelevant phase factor, so Alice's state $|\phi\rangle$ has already been deposited into Bob's qubit. If Alice measures the $|\Psi_{12}^{+}\rangle$ state, then Bob's state has been projected into $(-\alpha|0\rangle + \beta|1\rangle)$, which can be corrected by applying the σ_3 operator. Similarly, if Alice measures $|\Phi_{12}^{(-)}\rangle$ or $|\Phi_{12}^{(+)}\rangle$, Bob must apply σ_1 or $\sigma_3\sigma_1 = i\sigma_2$, respectively.

$$\{D^i\} = \{\mathbb{I}, \sigma_3, \sigma_1, i\sigma_2\}$$

If Bob applies the state transformation D^i based on Alice's measurement result $i \in [4]$, then he should always obtain a perfect replica of $|\phi\rangle$. Alice's two particles are left in the state Π^i, without any trace of the original state $|\phi\rangle$.

Note that if Bob applies a correction operator before receiving the classical message from Alice, he yields a random mixture of the four possible states. This is the maximally mixed state, which offers no information about $|\phi\rangle$.

2.1.2 Teleporting qudits

The qubit teleportation protocol from section 2.1 is perfect in terms of accurate quantum information transfer. Qubits are 2-dimensional quantum systems, while qudits are d-dimensional quantum systems. A perfect teleportation protocol for transfering information encoded in qudits is a straightforward generalization of the qubit protocol [15].

This protocol leverages a pair of N-state particles in a completely entangled state as the entangled resource.

$$\rho_{AB} = \sum_j \frac{|j\rangle \otimes |j\rangle}{\sqrt{N}}$$

As before, Alice performs a joint measurement on particles 1 and 2. One such measurement is the one whose eigenstates are

$$|\psi_{nm}\rangle = \sum_j \frac{e^{2\pi ijn/N}|j\rangle \otimes |(j+m)\text{mod}N\rangle}{\sqrt{N}}$$

Once Bob learns from Alice that she has obtained the result nm, he can apply the unitary correction operator

$$D^{nm} = \sum_k e^{2\pi ikn/N}|k\rangle \otimes \langle (k+m)\text{mod}N|$$

That transformation brings Bob's particle to the original state of Alice's particle 1, and the teleportation is complete.

2.1.3 Teleportation fidelity

The fidelity f of a teleportation protocol describes the overlap between the teleported state and the original state. We would like to maximize this quantity for reliable quantum communication.

A teleportation protocol $(\rho_{AB}, \{\Pi^i\}, \{D^i\})$ can teleport a state $\sigma_{C'}$ in quantum system C' into the particle(s) holding Bob's share of the entangled resource ρ_{AB} in quantum system C. Bob's decoding operation in the protocol can be noted as a quantum channel $D^i : B \to C$. Hence, the end-to-end teleportation protocol implements a quantum channel $\Lambda : C' \to C$. Let Φ^+

be a maximally entangled bipartite state of dimension $|C| \times |C|$. The noise in the teleportation channel Λ is typically measured using the entanglement fidelity [16]

$$F := \mathrm{tr}[\Phi^+(\mathrm{id} \otimes \Lambda)(\Phi^+)]$$

We are interested in average teleportation fidelity, which describes the ability of a teleportation protocol to accurately teleport any arbitrary unknown quantum state. The average teleportation fidelity $f := \int d\psi \langle\psi|\Lambda(\psi)|\psi\rangle$ is related to Entanglement fidelity F in the following way [16]

$$f = \frac{Fd+1}{d+1}$$

In contrast, the maximum teleportation fidelity without leveraging any nonclassical correlations is given by [17]

$$f_{cl} = \frac{2}{1+d}$$

Hence, a teleportation protocol requires states with $F > \frac{1}{d}$ in order to do better than classical techniques.

2.2 Dense coding

Dense coding is the dual problem to quantum teleportation. In a dense coding scheme, Bob encodes a classical message in a quantum state which Alice decodes with a measurement. The same elements of teleportation can be used to define a dense coding protocol:

- An entangled state, ρ_{BA}
- Encoding state transformation, $\{D^i\}$
- Decoding measurement, $\{\Pi^i\}$

Instead of simulating a quantum channel with 2-bits of classical information and a purely nonclassical resource state, we can simulate a 2-bit classical channel with a 1-qubit quantum channel.

Recall the standard teleportation protocol for qubits:

$$(\rho_{AB}, \{\Pi^i\}, \{D^i\}) = (|\Psi^{(-)}\rangle, \{|\Psi^{(-)}\rangle, |\Psi^{(+)}\rangle, |\Phi^{(-)}\rangle\, |\Phi^{(+)}\rangle\}, \{\mathbb{I}, \sigma_3, \sigma_1, i\sigma_2\})$$

Bob can encode classical information $i \in [4]$ by applying D^i to the B-part of $|\rho_B\rangle$. For example, $i = 4$ is encoded with $D^4 = i\sigma_2$ being applied to $|\rho_B\rangle$ so that the resultant entangled state becomes $\Pi^4 = |\Phi^{(+)}\rangle$. The joint state can then be measured with $\{\Pi^i\}$ to extract the encoded classical information. It is assumed that Bob and Alice share a noiseless quantum channel so that Alice can recieve Bob's share of the entangled resource towards the end of the protocol.

If the entangled resource utilized for dense coding is a d-dimensional completely entangled qudit state, then Bob can reliably encode a $2\log(d)$-bit classical message. This dense coding protocol uses the same data as the qudit teleportation protocol 2.1.2. Bob encodes the message $i \in [2\log(d)]$ by applying the corresponding encoding state transformation D^i into his system B of the shared state ρ_{AB}. Bob transmits his share of the entangled resource to Alice using a noiseless quantum channel, and Alice can decode the d-dimensional message by applying the measurement $\{\Pi^i\}$.

The classical channel resultant from dense coding is characterized by the transition probabilities

$$p(j|i) = \mathrm{tr}(\Pi^j_{AB}(\mathrm{id}_A \otimes D^j_B)\rho_{AB})$$

The quality of a dense coding scheme is characterized by its success probability

$$p_{\mathrm{succ}} = \frac{1}{N}\sum_{i=1}^{N} p(i|i)$$

This success probability is perfect for noiseless dense coding protocols using a maximally entangled resource. In noisy settings, the success probability is limited by the distillability of the shared entangled resource.

Success probability is connected to teleportation fidelity in the following way. Let $(\rho_{AB}, \{\Pi^i\}, \{D^i\})$ be a $|C|$-dimensional teleportation protocol. Then the following relation holds [16]:

$$F = |C|^{-2}\sum \mathrm{tr}(\Pi^i_{AC} w^i_{AC}) = N|C|^{-2}p_{\mathrm{succ}} \tag{2.1}$$

2.3 Badziag et al. states

The class of Badziag et al. states demonstrate that interactions between resource states and the local environment can enhance the fidelity of teleportation protocols [17]. A description of two-qubit states in this family follows.

Let $\mathcal{H}_A = \mathbb{C}^2 = \mathcal{H}_B$ and $x, y, p \in [0,1]$ be arbitrary. Consider the pure states

$$
\begin{aligned}
|\psi_1\rangle_{AB} &= \sqrt{x}|0\rangle_A \otimes |0\rangle_B - \sqrt{1-x}|1\rangle_A \otimes |1\rangle_B \\
|\psi_2\rangle_{AB} &= \sqrt{y}|0\rangle_A \otimes |1\rangle_B - \sqrt{1-y}|1\rangle_A \otimes |0\rangle_B.
\end{aligned}
$$

The resource state ρ_{AB} for our teleportation/dense coding protocol is a mixture of ψ_1 and ψ_2:

$$
\begin{aligned}
\rho_{AB} &= p|\psi_1\rangle\langle\psi_1|_{AB} + (1-p)|\psi_2\rangle\langle\psi_2|_{AB} \\
&= \begin{pmatrix} px & 0 & 0 & -pc_x \\ 0 & (1-p)y & -(1-p)c_y & 0 \\ 0 & -(1-p)c_y & (1-p)(1-y) & 0 \\ -pc_x & 0 & 0 & p(1-x), \end{pmatrix}
\end{aligned}
$$

where $c_x = \sqrt{x(1-x)}$ and $c_y = \sqrt{y(1-y)}$. This density matrix is purified by the following pure state $|\phi^\rho\rangle_{ABR}$ on AB and a third qubit R:

$$
\begin{aligned}
|\phi^\rho\rangle_{ABR} &= \sqrt{p}|\psi_1\rangle_{AB} \otimes |0\rangle_R + \sqrt{1-p}|\psi_2\rangle_{AB} \otimes |1\rangle_R \\
&= \left[\sqrt{px}, 0, 0, \sqrt{(1-p)y}, 0, -\sqrt{(1-p)(1-y)}, -\sqrt{p(1-x)}, 0\right]^T
\end{aligned}
$$

2.4 Werner states

Another resource state of interest for teleportation protocols is the family of Werner states [18]. These states emerge from depolarizing noise applied to one part of a maximally entangled bipartite quantum state. A Werner state is an incoherent combination of a pure maximally entangled state and completely mixed state. A description of qudit-Werner states follows.

Let $\mathcal{H}_A = \mathbb{C}^d = \mathcal{H}_B$ and $\{|0\rangle, |1\rangle, |2\rangle, \ldots, |d\rangle\}$ be the computational basis for a qudit. For $0 \leq i \leq j \leq d$ we define the states

$$|\psi_{ij}\rangle_{AB} = \frac{1}{\sqrt{d}}(|i\rangle_A|j\rangle_B + |j\rangle_A|i\rangle_B),$$

and for $0 \leq k < l \leq d$ we define

$$|\phi_{kl}\rangle_{AB} = \frac{1}{\sqrt{d}}(|k\rangle_A|l\rangle_B - |l\rangle_A|k\rangle_B).$$

The orthonormal sets of vectors

$$\begin{aligned} B_s &= \{|\psi_{ij}\rangle_{AB} \colon 0 \leq i \leq j \leq d\} \\ B_a &= \{|\phi_{kl}\rangle_{AB} \colon 0 \leq k < l \leq d\} \end{aligned}$$

span the *symmetric subspace* $\mathrm{Sym}^2(\mathbb{C}^d)$ and the *antisymmetric subspace* $\bigwedge^2(\mathbb{C}^d)$, respectively. The projectors Π_s and Π_a onto these spaces are given by

$$\begin{aligned} \Pi_s &= \sum_{\chi \in B_s} |\chi\rangle\langle\chi|_{AB} \\ \Pi_a &= \sum_{\chi \in B_a} |\chi\rangle\langle\chi|_{AB}. \end{aligned}$$

We have $\dim \Pi_s = \dim \mathrm{Sym}^2(\mathbb{C}^d) = \frac{d(d+1)}{2}$ and $\dim \Pi_a = \dim \bigwedge^2(\mathbb{C}^d) = \frac{d(d-1)}{2}$. The qudit-Werner state $W_{AB}(p)$ with $p \in [0,1]$ is defined as

$$W_{AB}(p) = \frac{2p}{d(d+1)}\Pi_s + \frac{2(1-p)}{d(d-1)}\Pi_a,$$

and purified for example by the pure state

$$|\phi_p^W\rangle = \sqrt{\frac{2p}{d(d+1)}} \sum_{|\chi\rangle \in B_s} |\chi\rangle \otimes |\chi\rangle + \sqrt{\frac{2(1-p)}{d(d-1)}} \sum_{|\chi\rangle \in B_a} |\chi\rangle \otimes |\chi\rangle.$$

2.5 Reduction criterion

A bipartite state ρ_{AB} can give rise to a nonclassical teleportation fidelity if and only if there is a locally processed version of ρ_{AB} violating the reduction criterion for seperability [16].

The reduction criterion is given by the following conditions

$$\rho_A \otimes \mathbb{I} - \rho_{AB} \geq 0, \mathbb{I} \otimes \rho_B - \rho_{AB} \geq 0$$

Note that the marginals of a Werner state are completely mixed:

$$\rho_A = \mathrm{Tr}_B(W_{AB}(p)) = \mathrm{Tr}_B\left(\frac{2p}{d(d+1)}\Pi_s + \frac{2(1-p)}{d(d-1)}\Pi_a\right) = \\ \frac{2p}{d(d+1)}\frac{d+1}{2}\mathbb{I}_A + \frac{2(1-p)}{d(d-1)}\frac{d-1}{2}\mathbb{I}_A = \frac{1}{d}\mathbb{I}_A.$$

Accordingly, the reduction criterion for a version of a Werner state which has been locally processed on only one side simplifies to

$$\frac{1}{d}\mathbb{I}_{AB} - \rho_{AB} \geq 0$$

Hence, a version of Werner state which has been locally processed on only one side violates the reduction criterion if its maximal eigenvalue is larger than $\frac{1}{d}$.

DESIGN

Our research focuses on finding good teleportation protocols for noisy quantum resources. A good teleportation protocol is one in which the teleported state approximates the original state as much as possible. Additionally, we provide a methodology for examining quantum states which may or may not be suitable resources for teleportation.

First, we describe the quantum circuit model. We will later optimize quantum circuits to maximize the fidelity of teleportation protocols and discover good quantum resources.

Quantum circuits describe a run of quantum computation. They are generally composed of wires, unitary operators, and measurements. A wire represents a unit of quantum computation (e.g., a qubit), unitary operators are state transformations, and measurement operators collapse the quantum state to return a classical result. The circuits are read from left to right. An example is given in figure 3.1.

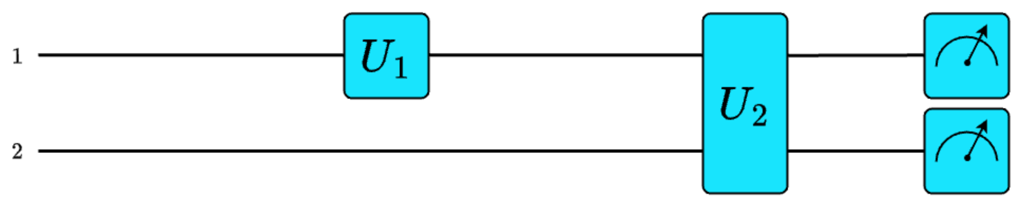


Figure 3.1: A quantum circuit with two quantum gates. The circuit qubits are initialized in the $|0\rangle$ state. The unitary U_1 is applied to qubit 1 and U_2 is applied to the joint state. At the end of the circuit, the qubits are measured. The circuit is read from left to right.

3.1 Optimizing teleportation protocols via dense coding

Variational quantum optimization parameterizes quantum circuit elements and optimizes them according to a cost function, similar to a neural net. We efficiently optimize quantum circuits for teleportation by exploiting the duality of teleportation and dense coding [16] to reduce circuit size.

An arbitrary teleportation protocol consists of an entangled state ρ_{AB}, encoding measurement $\{\Pi^i\}$, and decoding state transformation $\{D^i\}$. We optimize a teleportation protocol for any fixed resource state ρ_{AB}. Hence, the components of the protocol which must be parameterized for optimization are only Alice's encoding measurement $\{\Pi^i\}$ and Bob's decoding state transformation $\{D^i\}$. Figure 3.3 demonstrates the parameterized protocol in a practical and optimization-friendly manner. A key design choice for the optimization is fixing the teleporting state. The fidelity at which a teleportation protocol transmits half of a maximally entangled state is equivalent to the average fidelity of the protocol over all quantum states [19]. Hence, we can drastically reduce the state space of the optimization by forming a maximally entangled singlet state between an auxiliary wire and the wire to be teleported.

However, the circuit in figure 3.3 (a) is quite large and complex. An auxiliary wire is used to decrease the state space of exploration, but contributes to increasing the width of the circuit. The circuit can be simplified by exploiting the duality of teleportation and dense coding protocols [16]. The quantum circuit describing the same entangled state ρ_{AB}, now with Bob's encoding state transformation $\{D^i\}$ and Alice's decoding measurement $\{\Pi^i\}$, uses the same parameters from the teleportation protocol with a different interpretation. The dual circuit is shown in figure 3.3 (b) and can be easily optimized. In order to optimize over more general measurements (given by a POVM) and noisy decoding operations, we introduce additional wires that enable us to describe mixed states that are purified by the extra qubits. The dense coding circuit with auxiliary wires is shown in figure 4.1.

The fidelity of a teleportation protocol is a positive linear function of the success probability of its dual dense coding problem 2.1. Hence, we optimize the success probability of the dense coding circuit to indirectly optimize the teleportation protocol. The trained weights of the circuit in figure 4.1 map

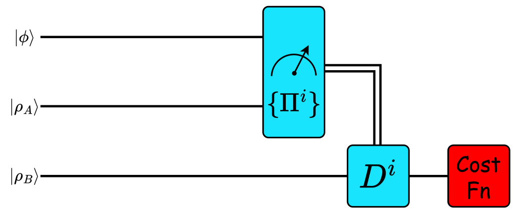


Figure 3.2: We can optimize the circuit above with a cost function that checks the overlap between $|\phi\rangle$ and the output of the bottom wire. However, the quality of a teleportation protocol is given by its average fidelity, that is the average overlap between an arbitrary state and the teleported state. Hence, this circuit requires high sampling, as we must sample a significant number of quantum states and optimize the teleportation on average for all of them.

to an optimized teleportation protocol for a noisy entangled resource.

3.2 Examination of qutrit-Werner states

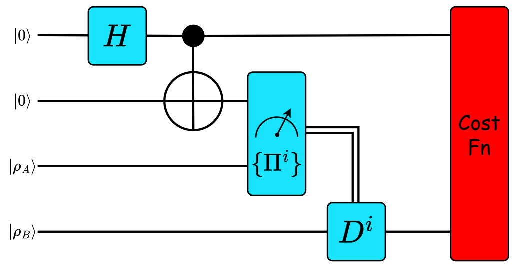
Werner states naturally emerge from depolarizing noise applied to one part of singlet state, a maximally entangled state. Werner states are a family of quantum states with $U \otimes U$ unitary symmetry that makes them useful to study entanglement, distillation and nonlocality. [18, 20]The usefulness of d-dimensional Werner states for telportation is not generally known.

In this section, we describe a methodology to study d-dimensional Werner states using qubit quantum circuit simulation. Operations such as 'ArbitraryUnitary' which are useful for circuit optimization are not yet supported for qutrit or qudit circuits. Hence, we encode qutrit-Werner states into a qubit circuit.

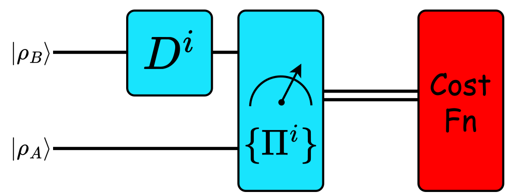
Consider the purification of an arbitrary qutrit-qutrit Werner state

$$|\phi_p^W\rangle = \sqrt{\frac{p}{6}} \sum_{|\chi\rangle \in B_s} |\chi\rangle \otimes |\chi\rangle + \sqrt{\frac{1-p}{3}} \sum_{|\chi\rangle \in B_a} |\chi\rangle \otimes |\chi\rangle$$

$|\phi_p^W\rangle$ is a sparse length-81 vector with non-zero values precisely in indices 0, 10, 12, 20, 24, 28, 30, 40, 50, 52, 56, 60, 68, 70, and 80.



(a) The fidelity of teleporting part of a singlet state is equal to the average fidelity of teleporting any state.



(b) This dense coding circuit has the same teleportation protocol parameters $(\rho_{AB}, \{\Pi^i\}, D^i)$ as (a). Optimizing for the success probability of this circuit is equivalent to optimizing (a).

Figure 3.3: Circuits that optimize the average fidelity of a teleportation protocol with reduced sampling compared to 3.2

We use the following encoding from 1 qutrit to 2 qubits.

$$\mathcal{F} : \boxed{\begin{array}{c} |0\rangle \rightarrow |00\rangle \\ |1\rangle \rightarrow |01\rangle \\ |2\rangle \rightarrow |11\rangle \end{array}}$$

Any encoding of the three qutrit states into three of four bases of two qubits is mathematically equivalent for optimization, but the choice can potentially impact convergence properties.

Given the qutrit state vector of $|\phi_p^W\rangle$, we keep the vector values and modify the indices (base 10) by the formula $\mathcal{F}$(index (base 3)) (base 10). For example, index 10 is first transformed to base 3 ($10_{10} \rightarrow 0101_3$), then transformed by $\mathcal{F}$ ($\mathcal{F}(0101) = 00010001$), and finally converted to base 10 ($00010001_2 = 17_{10}$). The qutrit-Werner state encoding into qubits is completed with the initialization of a length-256 vector with 15 non-zero values at indices 0, 17, 20, 51, 60, 65, 68, 85, 119, 125, 195, 204, 215, 221, 255.

Note that ququart-Werner states do not require reindexing because the

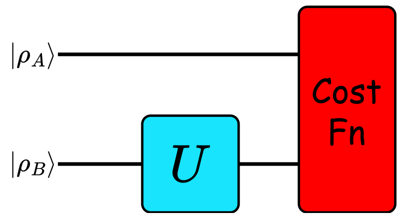


Figure 3.4: This circuit applies local processing to the Bob-side of a Werner state. A resource state state ρ_{AB} gives rise to a nonclassical teleportation fidelity if there is a locally processed version of itself violating the reduction criterion, $\rho_A \otimes \mathbb{I}_B - \rho_{AB} \geq 0$. A Werner state's marginals are completely mixed, so the reduction criterion simplifies to $\frac{1}{d}\mathbb{I}_{AB} - \rho_{AB} \geq 0$. We optimize the maximum eigenvalue of a Werner state with local processing on Bob's side. Any version of a Werner state which results from local processing on one side can be optimized for reduction criterion violation by increasing the maximal eigenvalue of the bipartite state.

dimensionality of two qubits is already equal to that of one ququart.

Given the embedding, we can define a quantum circuit to classify the usefulness of a d-level Werner state for teleportation. As shown in section 2.5, the cost function will be the maximal eigenvalue of Alice and Bob's final joint state. Surpassing a maximum eigenvalue of $\frac{1}{d}$ indicates violation of the reduction criterion, which indicates the resource state gives rise to nonclasical teleportation fidelity [16]. The corresponding quantum circuit for qutrit-Werner states is shown in figure 4.4.

EXPERIMENTS

4.1 Automatic optimization of teleportation protocols

We use Pennylane [21] and the QNETVO software package [22] to implement our optimization strategy. Specifically, we paramaterize operators and use gradient descent to maximize the fidelity of the protocol, seen in 4.1. Our coding framework, QNETVO, provides a way to describe a protocol in a quantum network and optimize over it.

We optimize the success probability of the dense coding protocol with and without additional auxiliary wires which represent a local environment. Non-classical teleportation advantages which beat the standard teleportation protocol are demonstrated in figures 4.2 and 4.3 with Badziag et al. resource states. Thanks to Hani Al Majed and Palak Kotwani for collecting these results with me in a past IBM-Illinois Discovery Accelerator project [23].

4.2 Usefulness of Werner states for teleportation

The Werner states are separable for $p > 0.5$, so we exclude these states. We explore various size local environments e and different sizes of Bob's output state b. The circuit used is developed in 2.4 and shown in 4.4. The results suggest that a quantum advantage for qutrit-Werner states in teleportation is limited to $p < \frac{2}{7}$, agreeing with [16]. The results for qutrit-Werner states are shown in figure 4.5. We also implement the optimization for ququart-Werner states and discover that these states yield non-classical teleportation results for $p < 0.1875$, shown in 4.6. Thanks to Yulie Arad for collecting these results with me in a current IBM-Illinois Discovery Accelerator project.

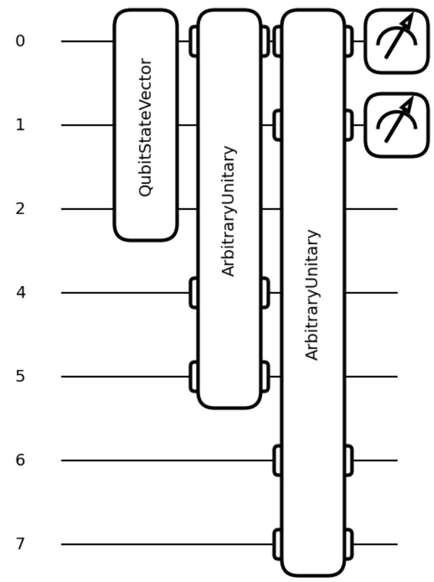


Figure 4.1: Dense coding circuit implementation in qNetVO. In order to optimize over more general measurements (given by a POVM) and noisy decoding operations, we introduce additional wires that enable us to describe mixed states that are purified by the extra qubits. Although we have a fully general decoding operation, our measurement requires two more auxiliary qubits to achieve full generality. As the number of auxiliary wires increases, the number of parameters in the circuit grows exponentially. Therefore, we set a cutoff as indicated above, as we can achieve fairly good results without requiring more advanced equipment than a standard laptop. In this case, the number of parameters is 318, which is much lower than the 4158 required for the fully general solution.

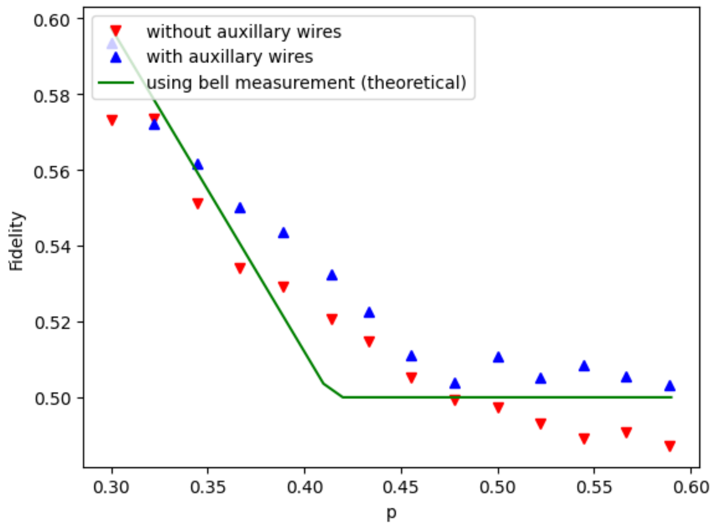


Figure 4.2: This plot for the Badziag et. al class of states with $x = 0$, $y = \frac{3-2\sqrt{2}}{4-2\sqrt{2}}$ reveals around $p = 0.5$ that we can surpass the optimal theoretical fidelity of a protocol using a Bell measurement with our optimized noisy ansatz.

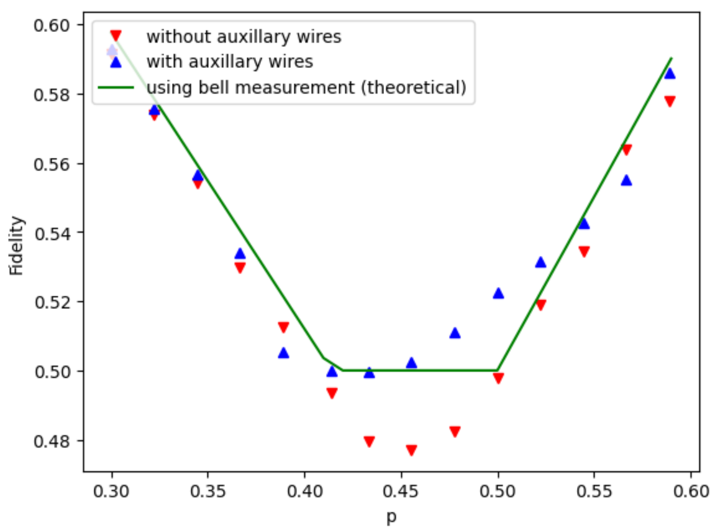


Figure 4.3: This plot for the the Badziag et. al class of states with $x = 1/2$, $y = \frac{3-2\sqrt{2}}{4-2\sqrt{2}}$ reveals around $p = 0.41$ that we can surpass the optimal fidelity of a protocol using a Bell measurement with both optimized noisy and noiseless ansätze.

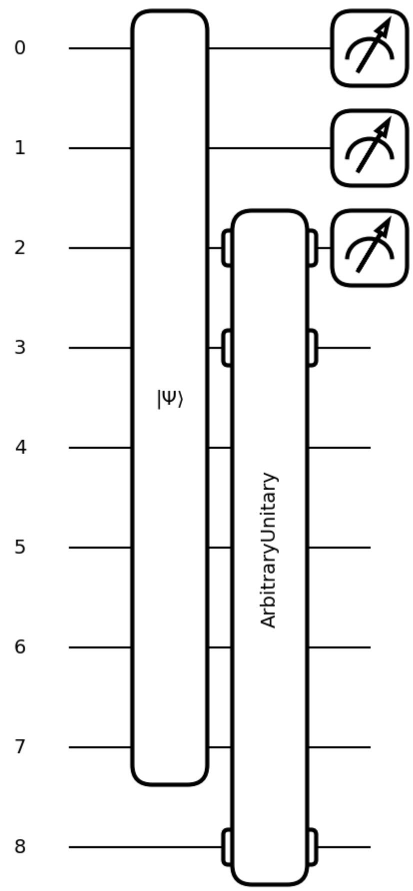
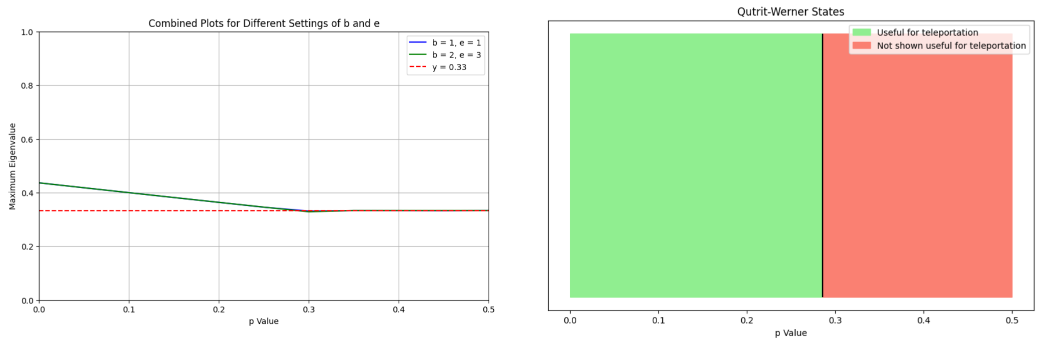
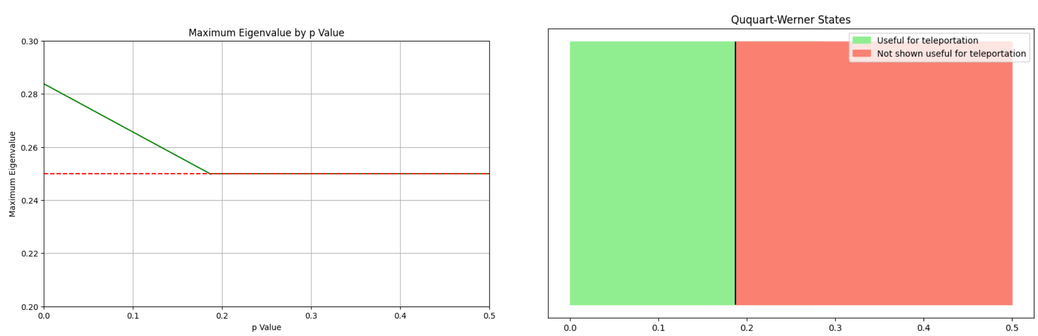


Figure 4.4: State initialization on wires 0 through 7 leaves an embedding of a qutrit-Werner state in wires 0 through 3. A parameterized unitary operation is applied to Bob's part of the bipartite state and a sufficient local environment. This circuit has the settings $e = 1$ and $b = 1$.



(a) (b)

Figure 4.5: Qutrit-Werner states demonstrate a non-classical avantage for teleportation when $p < \frac{2}{7}$. Adjusting the size of the local environment and Bob's output state impacts convergence but can be cleverly used to improve training speeds. Note that the different sets of data seem to be converging to the case of $b = 1$, $e = 1$, which was one of the least computationally expensive optimization settings due to a minimal number of auxiliary wires. We condense the fully general 4095 parameter solution into a 63 parameter solution with exactly the same results by reducing e and b to 1. These results reproduce previous work [16] with an automated technique that is easily extendable.



(a) (b)

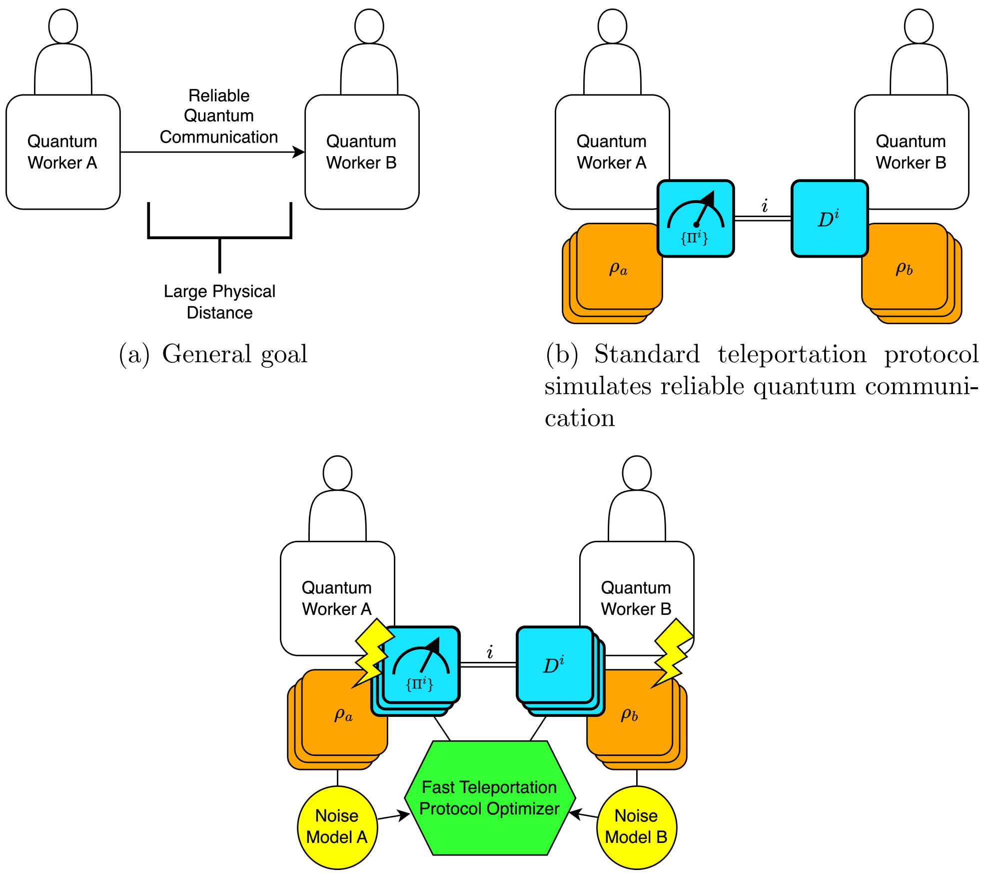
Figure 4.6: These plots show that quqart-Werner states are useful for teleportation for $p < 0.1875$. Again, a condensed 63 parameter solution (only 1 auxiliary wire and 1 output wire for Bob's state) suffices.

DISCUSSION

5.1 Significance of results

We show that qutrit-Werner states are useful for teleportation for $p < \frac{2}{7}$. This result agrees with other work that uses different techniques to come to the conclusion [16]. However, there is still room for increasing this upper bound. We only perform local processing on the B-side of the protocol. One might find that qutrit-Werner states are useful for teleportation for some $p > \frac{2}{7}$ by seeing a reduction criterion violation of a version of such a qutrit-Werner state with local processing on both parts of the bipartite state. Techniques used in this work must be modified, as the simplification of the reduction criterion to a maximal eigenvalue problem hinges on the marginal of the resource state being completely mixed, which is no longer true when both Alice and Bob perform local operations on their parts of the entangled resource. Similarly, it might still be possible to find useful ququart-Werner states for teleportation with $p > 0.1875$.

One particularly interesting result of this work is that the version of a Werner state generated by only using one auxillary environment wire and reducing the size of Bob's output state yields the same results as increasing the size of the environment and using Bob's full state in the maximum eigenvalue computation. Notably, reducing the size of the environment while keeping Bob's output size constant reduces the quality of the optimization. However, reducing the environment size along with Bob's system's output size yields exactly the right answer as the fully general solution. There is a massive speedup achieved by performing this trick. This may, however, just be a feature of Werner states or of doing local processing on only one side of the protocol.



(a) General goal

(b) Standard teleportation protocol simulates reliable quantum communication

(c) Reliable quantum communication with unreliable resources

Figure 5.1: Fast teleportation protocol optimizer

5.2 Future work

- Noise models for quantum computers are currently being developed [24]. Seeing this, it would be interesting to see quantum protocols being adapted in real time to account for the known noise. This idea is shown in figure 5.1.

- The stability of optimizing teleportation protocols can be further increased. The optimization of 3.3 is unstable and requires multiple shots to find non-classical teleportation protocols for a fixed resource state. If a fast teleportation protocol optimizer is implemented in practice, it would hopefully have better stability properties.

- Recent work has demonstrated the usefulness of a new teleportation protocol which improves teleportation fidelity by using ancillary entanglement catalytically (i.e., without depleting it) [25]. This work focused on optimizing teleportation protocols achieving the best teleportation fidelity when consuming a given entangled state. Future work might include optimizing teleportation protocols when consuming a given entangled state and using an arbitrary amount of entanglement catalytically.

- This work assumes that all quantum operations are of equal cost. However, this is not true in practice. In this setting, it would be interesting to explore integrating reinforcement learning techniques for optimization over a fixed gate set [26] or to optimize the protocol while considering the cost of different quantum operations.

- Note that the encoding measurement and decoding state transformation steps of teleportation can also introduce noise. Mitigating these noise sources in addition to the noise of the entangled resource would be an interesting line of work.

- It is possible to compute the maximal eigenvalue using a shallow quantum circuit [27]. Hence, future work can include examination of Werner states without evaluating the objective function classically.

- Other high-dimensional quantum states can also be examined using the current setup for Werner states. If their marginals are completely mixed, then the simple maximal eigenvalue optimization process continues to be effective for testing violations of the reduction criterion. Otherwise, different techniques to optimize for violations of the reduction criterion must be discovered and employed.

- This work only performs local processing on one part of Werner states to check for reduction criterion violations that show usefuless of these states for teleportation. It would be interesting to see results for reduction criterion violations of Werner states with local processing on both sides; however, the simplification of the reduction criterion to a maximal eigenvalue check falls through here because the marginals are no longer completely mixed 2.5. Performing local processing on both sides could reveal greater values of p for which qutrit-Werner states and ququart-Werner states are useful for teleportation.

CONCLUSION

This paper discusses efficient variational quantum optimization of teleportation protocols by exploiting duality with dense coding, as well as a shallow quantum circuit to test the capability of 3-level and 4-level Werner states as entangled resources in teleportation protocols.

The main results include demonstrations of surpassing the optimal theoretical fidelity of the standard teleportation protocol with custom quantum circuits for various Badziag et al. states, as well as a numerical demonstration that qutrit-Werner states provide a nonclassical advantage in teleportation when $p < \frac{2}{7}$ and ququart-Werner states provide the same when $p < 0.1875$.

Future work includes the testing of entangled states other than the class of Badziag et al. states and Werner states. Particularly, it is interesting to test other families of states which satisfy the reduction criterion but are not PPT.

CONTRIBUTIONS

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